GENERATING THE SURFACE MAPPING CLASS GROUP BY TWO ELEMENTS

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Abstract. Wajnryb proved in [W2] that the mapping class group of an orientable surface is generated by two elements. We prove that one of these generators can be taken as a Dehn twist. We also prove that the extended mapping class group is generated by two elements, again one of which is a Dehn twist. Another result we prove is that the mapping class groups are also generated by two elements of finite order.

1. Introduction

Let Σ be a compact connected oriented surface of genus g with one boundary component. We denote by Mod_g^1 the mapping class group of Σ, the group of isotopy classes of orientation-preserving diffeomorphisms $\Sigma \to \Sigma$ which restrict to the identity on the boundary. The isotopies are also required to fix the points on the boundary. If the diffeomorphisms and the isotopies are allowed to permute the points on the boundary of Σ, then we get the group $\mathrm{Mod}_{g,1}$. The extended mapping class group $\mathrm{Mod}_{g,1}^*$ of Σ is defined to be the group of isotopy classes of all (including orientation-reversing) diffeomorphisms of Σ. These three groups are related to each other as follows: $\mathrm{Mod}_{g,1}$ is contained in $\mathrm{Mod}_{g,1}^*$ as a subgroup of index two and the groups Mod_g^1 and $\mathrm{Mod}_{g,1}$ fit into a short exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Mod}_g^1 \to \mathrm{Mod}_{g,1} \to 1,$$

where $\mathbb{Z}$ is the subgroup of Mod_g^1 generated by the Dehn twist about a simple closed curve parallel to the boundary component of Σ.

In this paper, we will be interested in the groups Mod_g^1, $\mathrm{Mod}_{g,1}$ and $\mathrm{Mod}_{g,1}^*$. The mapping class group and the extended mapping class group of the closed surface of genus g obtained from Σ by gluing a disc along the boundary component are denoted by Mod_g and Mod_g^*.

The mapping class group is a central object in low-dimensional topology. Therefore, its algebraic structures are of interest. The problem of finding the generators for the mapping class group of a closed orientable surface was first considered by Dehn. He proved in [D] that Mod_g is generated by a finite set of Dehn twists. About quarter century later, Lickorish [L1, L2] also proved the same result, showing that $3g-1$ Dehn twists generate Mod_g. This

Date: February 1, 2008.
* Supported by TÜBA-GEBİP.

number was improved to $2g+1$ by Humphries [Hu]. These $2g+1$ generators are the Dehn twists about the curves $b, a_1, a_2, \ldots, a_{2g}$ in Figure 1, where the closed surface is obtained from Σ by gluing a disc along the boundary component. Humphries proved, moreover, that in fact the number $2g+1$ is minimal; i.e. Mod_g cannot be generated by $2g$ (or less) Dehn twists. Johnson [J] proved that the $2g+1$ Dehn twists about $b, a_1, a_2, \ldots, a_{2g}$ on Σ also generate Mod_g^1. Finally, the minimal number of generators for the mapping class group is determined by Wajnryb [W2]. He showed that Mod_g^1, and hence Mod_g, can be generated by two elements; one is a product of two Dehn twists (one is right and one is left) and the other is a product of $2g$ Dehn twists. Since the mapping class group is not abelian, the number two is minimal. Recently, it was shown by Brendle and Farb in [BF] that the mapping class group is generated by three torsion elements and by seven involutions.

Since $\mathrm{Mod}_{g,1}$ is a quotient of Mod_g^1, it is generated by the corresponding $2g+1$ Dehn twists. In order to generate the extended mapping class group $\mathrm{Mod}_{g,1}^*$, it suffices to add one more generator, namely the isotopy class of any orientation-reversing diffeomorphism.

In this paper we have three main results. First, we improve Wajnryb's result. We show that one of the two generators of Mod_g^1 can be taken as a Dehn twist. All Dehn twists involved in our generators are Dehn-Lickorish-Humphries generators. We also prove that the extended mapping class group $\mathrm{Mod}_{g,1}^*$ is generated by two elements, again one of which is a Dehn twist. Our proof is independent from that of Wajnryb [W2]. Next, we prove that the mapping class groups $\mathrm{Mod}_{g,1}$ and Mod_g are also generated by two torsion elements. Of course, this is not true for Mod_g^1 since it is torsion-free. In the last section of the paper, we transform the presentation of the mapping class group in [W3] into a presentation on our two generators.

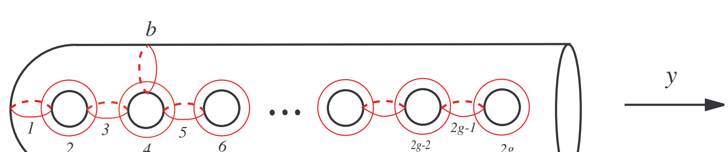

FIGURE 1. The curve labelled i is a_i.

2. PRELIMINARIES

Recall that if a is a simple closed curve on the oriented surface Σ, then the (right) Dehn twist A about a is the isotopy class of the diffeomorphism obtained by cutting Σ along a, twisting one of the side by 2π to the right and gluing two sides of a back to each other. We denote the curves by the letters

a, b, c, d (possibly with subscripts) and the Dehn twists about them by the corresponding capital letters A, B, C, D. Notationally we do not distinguish a diffeomorphism/curve and its isotopy class.

We use the functional notation for the composition of two diffeomorphisms; if F and G are two diffeomorphisms, then the composition FG means that G is applied first.

We define the curves c_i, d_i, $\bar{c}_i$ and $\bar{d}_i$ on Σ as shown in Figure 2, so that $\bar{c}_i$ (resp. $\bar{d}_i$) is obtained from c_i (resp. d_i) by rotating the surface Σ about the y-axis by π. These curves will be used through out the paper. Here, we assume that the surface of the paper is the yz-plane, the positive side of the x-axis is pointing above the page and the surface is invariant under the rotation by π about the y-axis.

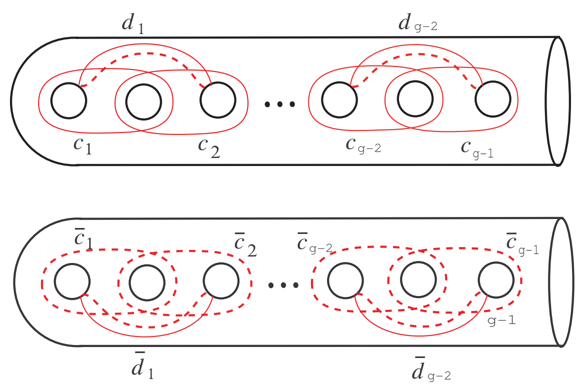

FIGURE 2. The curves $c_i, d_i, \bar{c}_i$ and $\bar{d}_i$.

Let G be a subgroup of Mod_g^1 or $\mathrm{Mod}_{g,1}^*$. Then G acts on the set of the isotopy classes of simple closed curves. If c is a simple closed curve, then we denote by c^G the G−orbit of c;

$$c^G = \{ F(c) : F \in G\}.$$

We record the following lemmas.

Lemma 1. *Let c be a simple closed curve on Σ, let F be a self-diffeomorphism of Σ and let $F(c) = d$. Then $FCF^{-1} = D^r$, where $r = \pm 1$ depending on whether F is orientation-preserving or orientation-reverving.*

Lemma 2. *Let c and d be two simple closed curves on Σ. If c is disjoint from d, then $CD = DC$.*

Lemma 3. *Let c and d be two simple closed curves on Σ. Suppose that $C \in G$ and the curve d is contained in c^G. Then D is also contained in G.*

Proof. Since $d \in c^G$, $F(c) = d$ for some $F \in G$. By Lemma 6, $FCF^{-1} = D^r$. This shows that D is contained in G. $\square$

3. The mapping class group Mod_g^1

Let S denote the product $A_{2g}A_{2g-1}\cdots A_2A_1$ of $2g$ Dehn twists in Mod_g^1 and let G be the subgroup of Mod_g^1 generated by B and S. We prove in this section that $G = \mathrm{Mod}_g^1$. It follows that the mapping class groups $\mathrm{Mod}_{g,1}$ and Mod_g are also generated by B and S. The main idea of the proof is to show that the G-orbit b^G of the curve b contains the simple closed curves $a_1, a_2, \dots, a_{2g}$.

Lemma 4. *The curves $c_1, c_2, \dots, c_{g-1}$ and $d_1, d_2, \dots, d_{g-2}$ of Figure 2 are contained in b^G, the G-orbit of b.*

Proof. This follows from these easily verified facts: The diffeomorphism S^{-1} maps

- b to c_1,
- c_i to d_i, and
- d_i to c_{i+1}. □

Remark. It can be shown that $S^{2g+1}(c_i) = \bar{c}_i$ and $S^{2g+1}(d_i) = \bar{d}_i$, so that $\bar{c}_i$ and $\bar{d}_i$ are also in b^G. These facts will be used in Section 5 in the proof of the fact that the mapping class groups $\mathrm{Mod}_{g,1}$ and Mod_g are generated by two torsion elements

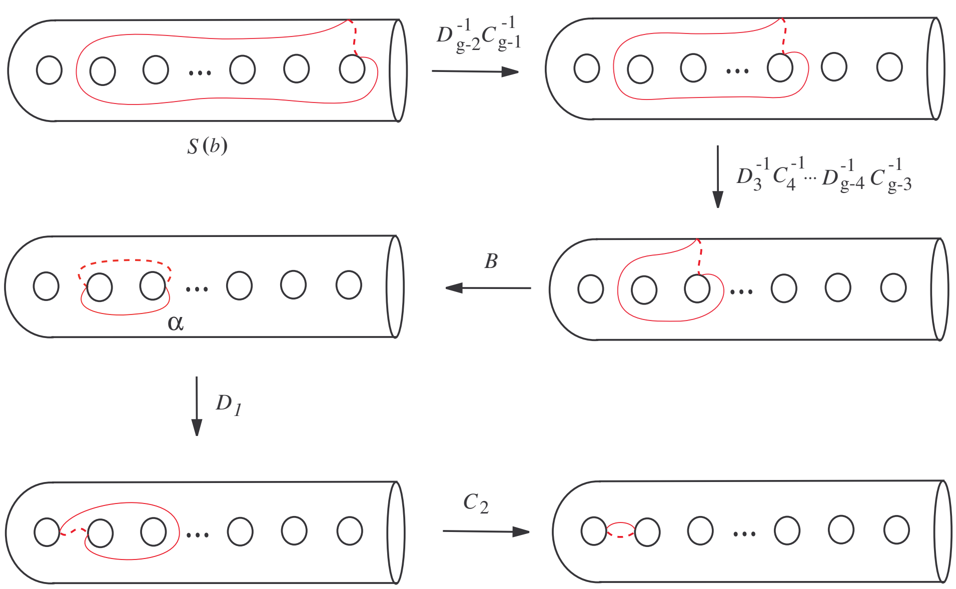

Figure 3. g is odd.

Theorem 5. *Suppose that $g \geq 2$. The subgroup G generated by B and S is equal to the mapping class group Mod_g^1.*

Proof. It can easily be shown that $S(a_i) = a_{i-1}$. Hence $SA_iS^{-1} = A_{i-1}$, and thus $A_{i-1} \in G$ if and only if $A_i \in G$.

Suppose that the curve a_{i_0} is contained in b^G for some i_0. Since $B \in G$, by Lemma 3 we get that A_{i_0} is also contained in G. Therefore, all A_i are contained in G. Since the mapping class group Mod_g^1 is generated by the $2g+1$ Dehn twists $A_1, A_2, \dots, A_{2g}$ and B, we conclude that the subgroup G is in fact equal to Mod_g^1. Therefore, in order to finish the proof of the theorem it suffices to show that a_i is contained in b^G for some i with $1 \leq i \leq 2g$.

Suppose first that g is odd. It is easy to see that the diffeomorphism

$$BD_3^{-1}C_4^{-1}D_5^{-1}C_6^{-1} \cdots D_{g-2}^{-1}C_{g-1}^{-1}$$

maps the curve $S(b)$ to a curve α and C_2D_1 maps α to a_3 (cf. Figure 3). Since all C_i and D_i are in G, the curve a_3 is contained in the G-orbit b^G of b.

Suppose now that g is even. In this case the diffeomorphism

$$BD_2^{-1}C_3^{-1}D_4^{-1}C_5^{-1} \cdots D_{g-2}^{-1}C_{g-1}^{-1}$$

maps the curve $S(b)$ to a_4. Again, since all C_i and D_i are in G, we conclude that a_4 is contained in b^G.

This concludes the proof of the theorem. □

4. The extended mapping class group $\mathrm{Mod}_{g,1}^*$

In this section we prove that the extended mapping class group $\mathrm{Mod}_{g,1}^*$ is also generated by two elements, one of which is a Dehn twist.

Consider the surface Σ embedded in the 3-space as shown in Figure 1. Let R denote the reflection across the xy-plane and let T denote the product $A_{2g}A_{2g-1} \cdots A_2A_1R$. Let H denote the subgroup of $\mathrm{Mod}_{g,1}^*$ generated by B and T. We prove that $H = \mathrm{Mod}_{g,1}^*$. Again, it follows that Mod_g^* is also generated by B and T. Recall that the H-orbit of the simple closed curve b is denoted by b^H.

Lemma 6. (i) *If g is even, then the curves $c_1, \bar{c}_2, c_3, \bar{c}_4, \dots, \bar{c}_{g-2}, c_{g-1}$ and $\bar{d}_1, d_2, \bar{d}_3, d_4, \dots, \bar{d}_{g-3}, d_{g-2}$ of Figure 2 are contained in b^H.*

(ii) *If g is odd, then the curves $c_1, \bar{c}_2, c_3, \bar{c}_4, \dots, c_{g-2}, \bar{c}_{g-1}$ and $\bar{d}_1, d_2, \bar{d}_3, d_4, \dots, d_{g-3}, \bar{d}_{g-2}$ of Figure 2 are contained in b^H.*

Proof. It can be shown easily that the diffeomorphism T^{-1} maps
- b to c_1,
- c_i to $\bar{d}_i$,
- $\bar{c}_i$ to d_i,
- d_i to c_{i+1}, and
- $\bar{d}_i$ to $\bar{c}_{i+1}$.

The lemma follows from these and Lemma 3. □

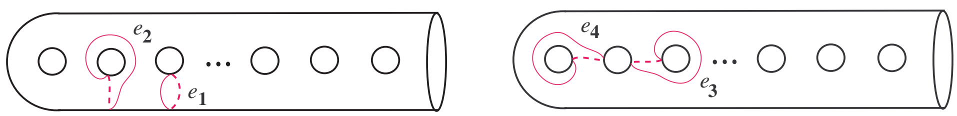

FIGURE 4. The curves e_i.

Lemma 7. *Suppose that g is odd. Then the curves e_1, e_2, e_3 and e_4 in Figure 4 are contained in b^H.*

Proof. Let U_i denote $(\bar{C}_i)^{-1}(\bar{D}_{i+1})^{-1}$. If i is even then U_i is contained in H by Lemma 6. Now the diffeomorphism

$$U_2U_4U_6 \cdots U_{g-3}(\bar{C}_{g-1})^{-1}T$$

is contained in H and it maps the curve b to e_1 (c.f. Figure 5). This proves that $e_1 \in b^H$.

For the proof of $e_2 \in b^H$ let U denote the diffeomorphism

$$\bar{C}_2D_2\bar{C}_4D_4 \cdots \bar{C}_{g-3}D_{g-3}.$$

Now it suffices to show that

$$C_1U\bar{C}_{g-1}T^{-1}$$

maps the curve $\bar{c}_{g-1}$ to e_2. This can be seen from Figure 6.

The curve e_3 is the image of the curve e_2 under the diffeomorphism $E_1^{-1}(\bar{C}_2)^{-1}$ (c.f. Figure 7). This proves that $e_3 \in b^H$.

Finally, $T^3(e_3) = e_4$ proving that $e_4 \in b^H$ (c.f. Figure 8). □

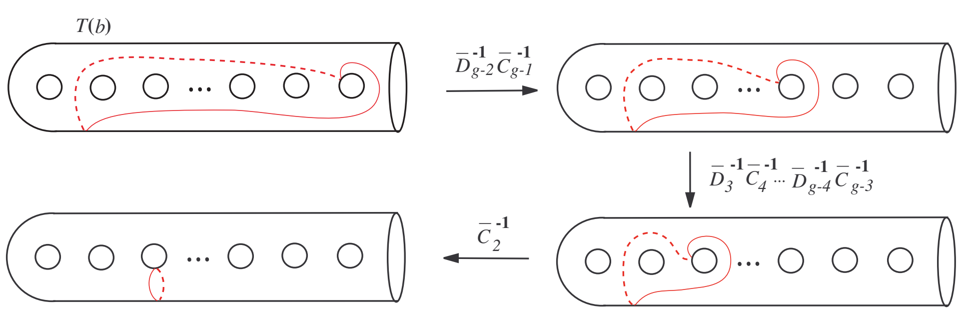

FIGURE 5. The proof of $e_1 \in b^H$.

Theorem 8. *The subgroup H generated by B and T is equal to the extended mapping class group $\mathrm{Mod}_{g,1}^*$.*

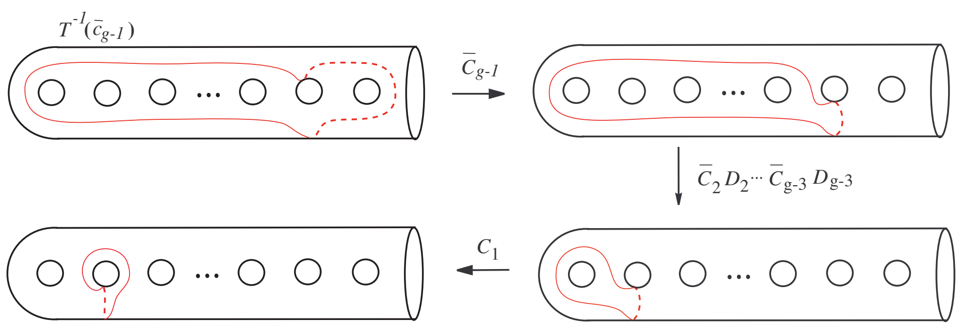

FIGURE 6. The proof of $e_2 \in b^H$.

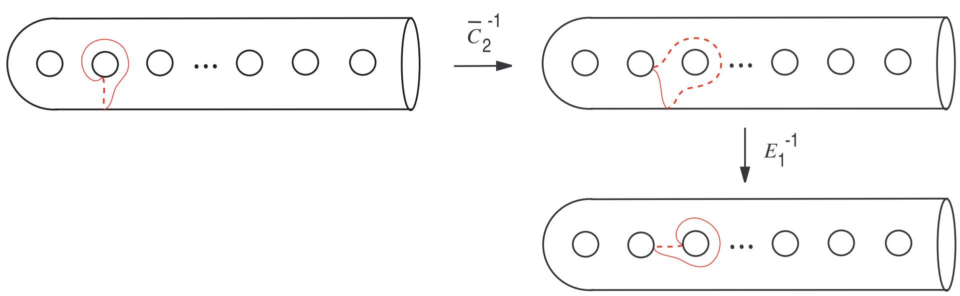

FIGURE 7. The proof of $e_3 \in b^H$.

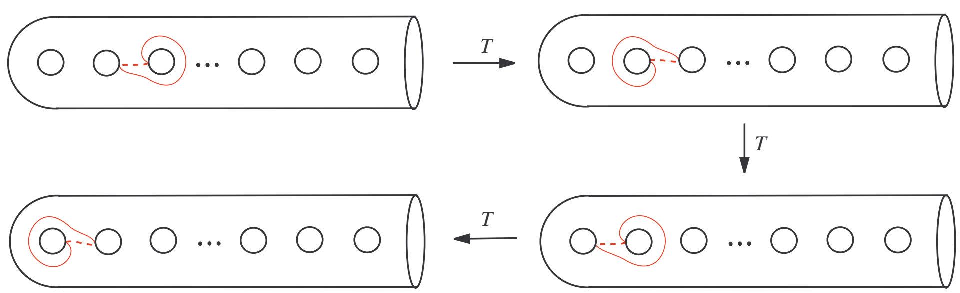

FIGURE 8. The proof of $e_4 \in b^H$.

Proof. We prove this theorem in the same way as Theorem 5; we show that the H-orbit b^H of the curve b contains simple closed curves $a_1, a_2, \ldots, a_{2g}$.

It is easy to show that $T(a_i) = a_{i-1}$ for $i = 2, 3, \ldots, 2g-1$. Hence, $TA_iT^{-1} = A_{i-1}$, and thus $A_i \in H$ if and only if $A_{i-1} \in H$.

Suppose that $a_{i_0} \in b^H$ for some i_0. Since B is contained in H, by Lemma 3 we get that A_{i_0} is also contained in H. Therefore, all A_i are contained in H.

Since $T \in H$, the reflection R is also contained in H. The extended mapping class group $\mathrm{Mod}^*_{g,1}$ is generated by the $2g+1$ Dehn twists $B, A_1, A_2, \ldots, A_{2g}$ and the reflection R. We conclude that the subgroup H is in fact equal to $\mathrm{Mod}^*_{g,1}$. Therefore, in order to finish the proof of the theorem it suffices to show that b^H contains a_i for some i with $1 \leq i \leq 2g$.

Suppose first that g is even. It follows from Lemma 3 and Lemma 6 that the diffeomorphism

$$V = C_1\bar{D}_1C_3\bar{D}_3 \cdots C_{g-3}\bar{D}_{g-3}C_{g-1}T^{-1}$$

is contained in H. Figure 9 shows that V maps the curve c_{g-1} to a_1. Since $c_{g-1} \in b^H$, a_1 is also in b^H. This finishes the proof in this case.

Suppose now that g is odd. It easy to verify that $(C_1)^{-1}E_4(e_2) = a_1$. Since $e_2 \in b^H$, and C_1 and E_4 are in H, we conclude that $a_1 \in b^H$.

This completes the proof of the theorem. $\square$

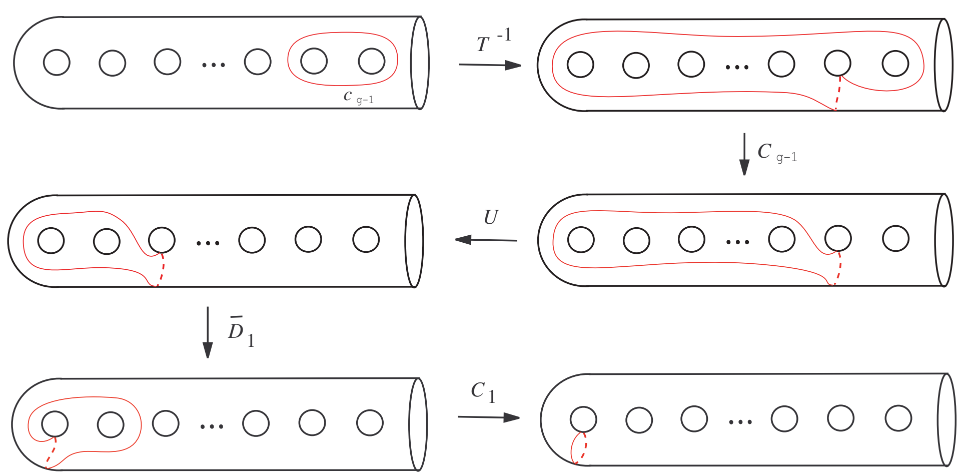

FIGURE 9. The proof of $a_1 \in b^H$ for even g.

5. Generating mapping class groups by two elements of finite order

In this section we prove that the mapping class groups $\mathrm{Mod}_{g,1}$ and Mod_g are generated by two elements of finite order. Clearly, this will be the minimum number of such generators. In their paper [BF], Brendle and Farb proved that these mapping class groups are generated by three torsion elements and asked if they can be generated by two. Therefore our result gives a positive answer to their question.

Let Σ be a surface with one boundary component as in Figure 1. In $\mathrm{Mod}_{g,1}$, let S denote the product $A_{2g}A_{2g-1}\cdots A_2A_1$. Note that S is of order $4g+2$. Throughout this section, let G denote the subgroup of $\mathrm{Mod}_{g,1}$

generated by the two torsion elements S and BSB^{-1}. We prove that $G = \mathrm{Mod}_{g,1}$ for $g \geq 3$.

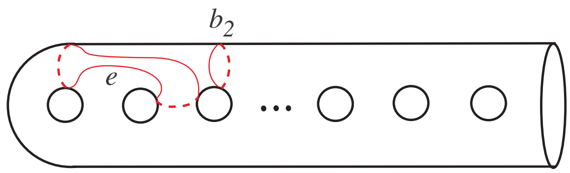

FIGURE 10. The curves e and b_2.

In the proof of the main result of this section, we use the celebrated lantern relation, which was discovered by Dehn and redisovered by Johnson. This relation is read as follows;

$$A_1 A_3 A_5 B_2 = B D_1 E, \tag{1}$$

where the B_2 and E are the Dehn twists about the curves b_2 and e in Figure 10. We rewrite the relation as

$$A_1 = (BA_3^{-1})(D_1 A_5^{-1})(EB_2^{-1}). \tag{2}$$

The strategy of the proof is to show that the statements inside each paranthesis are in G. Then the rest of the proof is easy.

Let us define $\mathcal{B}$ as the subset of nonseparating simple closed curves consisting of those curves x such that $XB^{-1} \in G$. That is,

$$\mathcal{B} = \{\ x \mid x \text{ is a nonseparating simple closed curve and } XB^{-1} \in G\}.$$

We first state the following easy to prove, but very useful, lemmas.

Lemma 9. *Suppose that two simple closed curves x and y are contained in $\mathcal{B}$. Then XY^{-1} is contained in the subgroup G.*

Lemma 10. *Suppose that $y \in \mathcal{B}$ and $XY^{-1} \in G$. Then $x \in \mathcal{B}$.*

Lemma 11. *Suppose that a curve y is contained in $\mathcal{B}$ and x is in the $\langle S \rangle$-orbit of y. Then x is contained in $\mathcal{B}$.*

Proof. By assumption, there is an integer k such that $x = S^k(y)$. Since the element

$$XY^{-1} = (S^k Y S^{-k})Y^{-1} = S^k(YB^{-1})(BSB^{-1})^{-k}(BY^{-1})$$

is contained in G, the lemma is proved. □

Corollary 12. *The curves $c_i, \bar{c}_i, d_j, \bar{d}_j, S(b)$ are contained in $\mathcal{B}$ for all $i = 1, 2, \ldots, g-1$ and $j = 1, 2, \ldots, g-2$.*

Proof. The fact that c_i, d_j are in the $\langle S \rangle$-orbit of b is shown in Section 3. It can be shown that S^{2g+1} is isotopic to the rotation by π about the y-axis. Since $\bar{c}_i = S^{2g+1}(c_i)$ and $\bar{d}_j = S^{2g+1}(d_j)$, the corollary follows from Lemma 11. □

Lemma 13. *Suppose that $g \geq 3$. Then for each $i = 1, 2, \ldots, g$, the curves a_i, b_2 and e are contained in $\mathcal{B}$.*

Proof. We prove first that the curves a_i are contained in $\mathcal{B}$. Since all a_i are contained in the same $\langle S \rangle$-orbit, it is enough to prove that any one of a_i is in $\mathcal{B}$.

Suppose that g is even, so that $g \geq 4$. Let V denote the product $C_3^{-1} D_4^{-1} \cdots D_{g-4}^{-1} C_{g-3}^{-1} D_{g-2}^{-1} C_{g-1}^{-1}$. We have shown in the proof of Theorem 5 that the diffeomorphism $BD_2^{-1}VS$ maps the curve b to a_4. Since the curve $\bar{c}_1$ is disjoint from each c_i and d_j for $i \geq 1$, $j \geq 3$, the Dehn twist $\bar{C}_1$ about $\bar{c}_1$ commutes with each C_i and D_j. Therefore it commutes with V. Let $x = VS(b)$. Since $\bar{C}_1 C_i^{-1} \in G$ and $\bar{C}_1 D_j^{-1} \in G$, we have $\bar{C}_1^{g-3} V \in G$. By the above lemmas, we also have $SBS^{-1}\bar{C}_1^{-1} \in G$. Therefore

$$
\begin{aligned}
(3) \qquad (\bar{C}_1^{g-3}V)(SBS^{-1}\bar{C}_1^{-1})(V^{-1}\bar{C}_1^{3-g}) &= (VS)B(VS)^{-1}\bar{C}_1^{-1} \\
(4) \qquad &= X\bar{C}_1^{-1}
\end{aligned}
$$

is contained in G. Therefore, $x \in \mathcal{B}$. Moreover, since $BD_2^{-1} \in G$ and $XD_2^{-1} \in G$, we obtain

$$(BD_2^{-1})(XD_2^{-1})(D_2B^{-1}) = (BD_2^{-1})X(BD_2^{-1})^{-1}D_2^{-1} = A_4D_2^{-1}$$

is contained in G. This shows that a_4, and hence, all a_i, is in $\mathcal{B}$.

Suppose next that g is odd and $g \geq 5$. Now again let V denote the diffeomorphism $D_3^{-1} C_4^{-1} D_5^{-1} C_6^{-1} \cdots D_{g-2}^{-1} C_{g-1}^{-1}$. We have shown in the proof of Theorem 5 that the diffeomorphism $C_2 D_1 BVS$ maps the the curve b to a_3. In this case, let x denote the curve $VS(b)$, as above. The equation (3) shows that $x \in \mathcal{B}$. We now use the equality

$$(C_2D_1BC_4^{-3})(XC_4^{-1})(C_2D_1BC_4^{-3})^{-1} = A_3C_4^{-1}$$

to conclude that a_3 is in $\mathcal{B}$.

Suppose now that $g = 3$. It can easily be shown that $C_1^{-1}D_1^{-1}S^2BS(b) = a_5$. We use this fact to prove that a_5 is contained in $\mathcal{B}$. Notice that $C_2^{-1}S(b) = b_2$. The equation

$$(\bar{C}_1C_2^{-1})(SBS^{-1}C_2^{-1})(C_2\bar{C}_1^{-1}) = (C_2^{-1}S)B(S^{-1}C_2)C_2^{-1} = B_2C_2^{-1}$$

shows that $b_2 \in \mathcal{B}$. Let y denote the curve $BS(b)$. From the equation

$$(B\bar{C}_1^{-1})(SBS^{-1}B_2^{-1})(B\bar{C}_1^{-1})^{-1} = YB_2^{-1},$$

we conclude that $y \in \mathcal{B}$. By Lemma 11, $z = S^2(y) = S^2BS(b)$ is also contained in $\mathcal{B}$. Finally, the fact that a_5 is in $\mathcal{B}$ follows from the equation

$$(C_1^{-1}D_1^{-1}B_2^2)(ZB_2^{-1})(B_2^{-2}D_1C_1) = A_5B_2^{-1}.$$

This concludes the proof of $a_i \in \mathcal{B}$. In order to finish the proof of the lemma, it remains to prove that b_2 and e are contained in $\mathcal{B}$.

It is easy to see that $C_2^{-1}A_6A_5A_4(b) = b_2$ and $A_2A_1A_4^{-1}C_1(a_5) = e$. It can be shown that the diffeomorphisms $A_1^{-2}C_2^{-1}A_6A_5A_4$ and $B_2^{-2}A_2A_1A_4^{-1}C_1$ are in G. Finally, $b_2 \in G$ follows from

$$(A_1^{-2}C_2^{-1}A_6A_5A_4)(BA_1^{-1})(A_1^{-2}C_2^{-1}A_6A_5A_4)^{-1} = B_2A_1^{-1},$$

and $e \in G$ follows from

$$(B_2^{-2}A_2A_1A_4^{-1}C_1)A_5B_2^{-1}(B_2^{-2}A_2A_1A_4^{-1}C_1)^{-1} = EB_2^{-1}.$$

This completes the proof. □

Theorem 14. *The mapping class group* $\mathrm{Mod}_{g,1}$ (*and hence* Mod_g) *is generated by two elements of finite order.*

Proof. If $g = 1$ then A_2A_1 and $A_1A_2A_1$ are of orders 6 and 4, respectively, and they generate $\mathrm{Mod}_{1,1}$. If $g = 2$ then $A_4A_3A_2A_1$ and $A_5A_4A_3A_2A_1$ are of orders 10 and 6, respectively, and they generate $\mathrm{Mod}_{2,1}$. Suppose that $g \geq 3$ and let G be the subgroup of $\mathrm{Mod}_{g,1}$ generated by the elements $S = A_{2g}A_{2g-1}\cdots A_2A_1$ and BSB^{-1}, which are both of order $4g+2$. Since the curves a_3, a_5, b, b_2, d_1 and e are all contained in $\mathcal{B}$, the elements BA_3^{-1}, $D_1A_5^{-1}$ and EB_2^{-1} are contained in G. By the lantern relation (2), $A_1 \in G$. Therefore, $S^{-i+1}A_1S^{i-1} = A_i \in G$.

Finally, the element $C_1^{-1}A_1^{-1}A_4A_3$ is in G and maps a_2 to b. Since $A_2 \in G$, this shows that $B \in G$. Consequently, $G = \mathrm{Mod}_{g,1}$.

This finishes the proof of the theorem. □

6. A presentation of the mapping class group on two generators

In this last section we transform the Wajnryb presentation of the mapping class group Mod_g^1 to a presentatoin on the two generators B and S. It turns out that the number of relations in the new presentation depends linearly on g, whereas it is quadratic in the original presentation.

Theorem 15. ([W1, W3]) *Let* $g \geq 3$. *The mapping class group* Mod_g^1 *admits a presentation with generators* $B, A_1, A_2, \ldots, A_{2g}$ *and with defining relations*

(i) $BA_i = A_iB$ *if* $i \neq 4$, *and* $A_jA_k = A_kA_j$ *if* $|j-k| \geq 2$,

(ii) $BA_4B = A_4BA_4$, $A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}$ *for* $1 \leq i \leq 2g-1$,

(iii) $(A_1A_2A_3)^4 = B(A_4A_3A_2A_1^2A_2A_3A_4)B(A_4A_3A_2A_1^2A_2A_3A_4)^{-1}$,

(iv) $A_1A_3A_5wBw^{-1} = (t_2t_1)^{-1}B(t_2t_1)t_2^{-1}Bt_2B$,

where
$t_1 = A_2A_1A_3A_2$, $t_2 = A_4A_3A_5A_4$ *and*
$w = A_6A_5A_4A_3A_2(t_2A_6A_5)^{-1}B(t_2A_6A_5)(A_4A_3A_2A_1)^{-1}$.

Suppose that $g \geq 3$. We have shown above that the curve b is mapped to a_3 by the diffeomorphism

$$X_1 = C_2 D_1 B D_3^{-1} C_4^{-1} D_5^{-1} C_6^{-1} \cdots D_{g-2}^{-1} C_{g-1}^{-1} S \tag{5}$$

if g is odd, and to a_4 by

$$X_2 = B D_2^{-1} C_3^{-1} D_4^{-1} C_5^{-1} \cdots D_{g-2}^{-1} C_{g-1}^{-1} S \tag{6}$$

if g is even.

From the equalities $S^{-2k+1}(b) = c_k$ and $S^{-2k}(b) = d_k$ we get that

$$S^{-2k+1} B S^{2k-1} = C_k$$

and

$$S^{-2k} B S^{2k} = D_k,$$

which should replace C_k and D_k in the equations (5) and (6). Also note that $S(a_k) = a_{k-1}$ and hence $S A_k S^{-1} = A_{k-1}$. Let us now define

$$X = \begin{cases} S^3 X_1 & \text{if } g \text{ is odd,} \\ S^4 X_2 & \text{if } g \text{ is even,} \end{cases} \tag{7}$$

so that $S^{-k} X(b) = a_k$, and therefore

$$A_k = S^{-k} X B X^{-1} S^k = B^{S^{-k} X} \tag{8}$$

for every $k = 1, 2, \ldots, 2g$.

The presentation of the mapping class group Mod_g^1 in Theorem 15 has $2g+1$ generators and $2g^2+g+2$ relations. By replacing each A_k in Theorem 15 by $S^{-k} X B X^{-1} S^k$, we get another presentation of Mod_g^1 on the two generators B and S. Now these relations are not as nice as those of Wajnryb's. But, the number of relations in the new presentation reduces to $4g + 1$.

For any i, j with $|i - j| > 1$ and for any k, all the relations $A_{i+k} A_{j+k} = A_{j+k} A_{i+k}$ reduce to the single relation $B^X B^{S^{i-j} X} = B^{S^{i-j} X} B^X$. This shows that the relations in (i) gives rise to $(2g - 1) + (2g - 2) = 4g - 3$ relations.

Each braid relation $A_k A_{k+1} A_k = A_{k+1} A_k A_{k+1}$ reduces to the single relation $B^X B^{S^{-1} X} B^X = B^{S^{-1} X} B^X B^{S^{-1} X}$. Thus the relations in (ii) reduces to two relations.

Consequently the new presentation of Mod_g^1 is given as follows: Let P denote B^X and let U denote S^{-1}, where X is defined as in (7).

Theorem 16. *Let $g \geq 3$. The mapping class group Mod_g^1 admits a presentation with generators B and U, and with defining relations*

(i) $B P^{U^i} = P^{U^i} B$, *for* $1 \leq i \leq 2g$ *and* $i \neq 4$,

(ii) $P P^{U^i} = P^{U^i} P$ *for* $2 \leq i \leq 2g - 1$,

(iii) $B P^{U^4} B = P^{U^4} B P^{U^4}$ *and* $P P^U P = P^U P P^U$,

(iv) $(P^U P^{U^2} P^{U^3})^4 = B B^V$, *and*

(v) $P^U P^{U^3} P^{U^5} B^W = B B^{t_1^{-1} t_2^{-1}} B^{t_2^{-1}}$,

where

$V = P^{U^4} P^{U^3} P^{U^2} P^{U^1} P^{U^1} P^{U^2} P^{U^3} P^{U^4}$.

$t_1 = P^{U^2} P^U P^{U^3} P^{U^2}$, $t_2 = P^{U^4} P^{U^3} P^{U^5} P^{U^4}$ *and*
$W = P^{U^6} P^{U^5} P^{U^4} P^{U^3} P^{U^2} (t_2 P^{U^6} P^{U^5})^{-1} B (t_2 P^{U^6} P^{U^5})(P^{U^4} P^{U^3} P^{U^2} P^U)^{-1}$.

The presentation of the mapping class group Mod_g is obtained from that of Mod_g^1 by adding one more relation (c.f. [W3]). Hence, in the new presentation on the generators B and S the number of relation reduces to $4g+2$.

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